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MINIFICATION PROCESSES

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# MINIFICATION PROCESSES

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## Abstract

It is shown that the autoregressive, Markovian minification processes introduced by Tavares and Sim can be extended to marginal distributions other than the exponential and Weibull distributions. Necessary and sufficient conditions on the hazard rate of the marginal distributions are given for a minification process to exist. Results are given for the derivation of the autocorrelation function; these correct the expression for the Weibull given by Sim. Monotonic transformations of the minification processes are also discussed and generate a whole new class of autoregressive processes with fixed marginal distributions. Processes generated by a maximum operation are also introduced and a comparison of three different Markovian processes with uniform marginal distributions are given.

**Keywords:** Minification; Time series

## 1 Introduction

In a series of papers Tavares(1977,1980a,b) introduced two stationary Markov processes with similar structural form which he had found useful in hydrological applications. The first process, a maximum process, has an extreme value or Gumbel's marginal distribution, for which the density function is

$$f_X(x) = \beta e^{-x\beta} e^{-e^{-x\beta}} \quad \beta \geq 0; -\infty < x < \infty \quad (1)$$

and the hazard rate is

$$\lambda_X(x) = f_X(x)/S_X(x) = \beta e^{-x\beta} e^{-e^{x\beta}} / (1 - e^{-e^{-x\beta}}) \quad (2)$$

where  $S_X(x) = 1 - F_X(x) = P(X > x)$  is the survivor function. The second process, a minimum or minification process, has negative exponential marginal distribution for which

$$f_X(x) = \beta e^{-\beta x} \quad x \geq 0, \beta \geq 0 \quad (3)$$

and

$$\lambda_X(x) = \beta. \quad (4)$$

This second process is also investigated by Daley, Chernick and Littlejohn(1988) who demonstrate some very interesting theoretical properties, in particular that the process is a time reversed version of the linear, additive EAR(1) model of Gaver and Lewis(1980). Further, Sim(1986) has shown that the structural form of this minification process of Tavares will also accommodate a Weibull( $\kappa, \beta$ ) marginal distribution. The Weibull ( $\kappa, \beta$ ) density function is

$$f_X(x) = \kappa \beta^\kappa x^{\kappa-1} e^{-(\beta x)^\kappa} \quad x \geq 0; \kappa > 0; \beta > 0 \quad (5)$$

and

$$\lambda_X(x) = \kappa \beta^\kappa x^{\kappa-1}. \quad (6)$$

Thus the hazard rate is a power law function, decreasing from infinity to zero as  $x$  increases when  $\kappa < 1$ , and increasing monotonically from zero to infinity as  $x$  increases when  $\kappa > 1$ . For  $\kappa = 1$  this is an exponential distribution.

This second process is called a minification process because the observations  $\{X_n\}$ , where the  $X_n$ 's are positive valued, are generated by the equation

$$X_n = K \min(X_n, Z_{n-1}). \quad (7)$$

Here  $K > 1$  is a constant, and  $\{Z_n\}$  is an innovation process of independent and identically distributed random variables chosen to ensure that  $\{X_n\}$  is a

stationary Markov process with marginal distribution function  $F_X(x)$ . In the negative exponential case (3) it is found that the autocorrelation function of the process has the familiar geometrically decreasing form of an autoregressive process of order one, AR(1), i.e.,  $\rho_X(j) = \rho^j$ ,  $j = 1, 2, \dots$ .

The purpose of this work is to explore the generality of (7) in two distinct ways. First we attempt to determine the range of possible marginal distribution functions  $F_X(x)$  for non-negative  $X_n$ . We shall see that it is possible to specify fairly simple conditions in terms of the hazard function of the distribution which determine whether that distribution may be used as a marginal distribution in (7). The second approach to this investigation arises from the fact that the structure of (7) is so simple that many of the important features of the process may be invariant under instantaneous monotone transformation. In this way, we may derive simple Markov processes based on (7) but with any marginal distribution we wish, even if it is one for which a minimization process does not exist. Moreover we may also immediately deduce many of the basic properties of the resulting process directly from our results.

Sequences of non-negative random variables find applications in many fields. The work of Tavares was motivated by hydrological considerations, for example modelling of run-off data. This data tends to have long tails and thus cannot be modelled by exponential processes such as the random linear coefficient EAR(1) processes of Gaver and Lewis (1980). (See Lewis, 1985, for a summary of these models). Weibull or extreme-value random variables are commonly used for modelling the marginal distribution functions of run-off series, but processes with these marginal distributions cannot be generated with linear random coefficient models. Thus the minimization processs are important as a source of time series for such processes.

Another case of interest is time series of wind velocity magnitudes. These again are positive valued random variables and their simulation is important for driving, for example, models of temperature mixing in the ocean. Brown,

Katz, and Murphy (1984) note that although studies have shown that Weibull marginal distribution have been found adequate for wind velocity magnitudes, unfortunately, "no time series models have been rigorously developed for random variables possessing a Weibull distribution ." They therefore resort to transformations of the data. Wind power data, being the square of wind velocity data, is even more likely to need very long tailed marginal distributions.

Again, in reliability studies, sequences of times-between-failures are correlated and models are chosen on the basis of a generally non-constant marginal hazard rate. We show how it is possible to generate such sequences with minification processes, for example with the familiar 'bath-tub' hazard rate. An analytical representation for this type of hazard rate is given in Gaver and Acar(1979).

## 2 The General Minification Process.

For the moment we make no assumptions about the marginal distribution of  $\{X_n\}$  or of  $\{Z_n\}$  save only to assume that a distribution can be found for  $Z_n$  so that  $\{X_n\}$  is a stationary Markov process given by ( 7 ) . Suppose now that the survivor function of the *non-negative valued* random variable  $X_n$  is given by  $S_X(x) = P(X > x)$ . It is easily verified from ( 7 ) that the survivor function of  $Z_n$  must satisfy

$$S_Z(x) = S_X(Kx)/S_X(x) \quad x \geq 0; K > 1 \quad (8)$$

This shows that  $K$  must be greater than one; otherwise, since  $S_X(x)$  is generally decreasing, the function  $S_Z(x)$  would be greater than one in value for some  $x$ .

Note that for the general process we may write the survivor function  $S_X(x)$  in terms of the cumulative hazard  $\Lambda_X(x)$ , or the hazard rate,  $\lambda_X(x)$ , thus:  $S_X(x) = \exp[-\Lambda_X(x)] = \exp[-\int_0^x \lambda_X(t)dt]$ . Thus, equation ( 8 ) may be recast in terms of hazard functions as follows:

$$\Lambda_Z(x) = \int_0^x \lambda_Z(t)dt = \int_x^{Kx} \lambda_X(t)dt \quad x \geq 0 \quad (9)$$

We now consider what is the set of possible marginal distributions for  $X_n$  in the general minification process (7). Clearly, a necessary and sufficient condition for a distribution to be suitable for this purpose is that the right hand side of (8) is a survivor function, or equivalently that the right hand side of (9) is a cumulative hazard function. In the latter case, we require that  $-\ln S_Z(x) = \Lambda_Z(x) = \int_x^{Kx} \lambda_X(t)dt$  be a non-decreasing function of  $x$ , for all  $x \geq 0$ , and be increasing for some  $x > 0$ . This is equivalent to

$$\int_y^x \{K\lambda_X(Ks) - \lambda_X(s)\} ds \geq 0$$

for all  $x \geq y \geq 0$ , and positive for some  $x \geq y \geq 0$ , which reduces to  $K\lambda_X(Ks) \geq \lambda_X(s)$  for all  $s \geq 0$ , with inequality for some  $s \geq 0$ . Multiplying by  $s$  yields the alternative necessary and sufficient condition that

$$x\lambda_X(x) \quad (10)$$

is a non-decreasing function of  $x$ , for all  $x \geq 0$ , and an increasing one for some  $x \geq 0$ .

Now assume that (10) is true. Then as  $x \rightarrow \infty$  either (i)  $S_Z(x) \rightarrow 0$  or (ii)  $S_Z(x) \rightarrow p$ , where  $p \in (0, 1)$ .

In case (i)  $S_Z(x)$  is a proper survivor function and so the minification process (7) can be constructed with the required marginal distribution for  $\{X_n\}$  by using an innovation process  $\{Z_n\}$  whose survivor function is defined by (8).

In case (ii),  $S_Z(x)$  is not strictly a proper survivor function having, in effect, an atom of probability  $p$  located at infinity. Such a property would seem to rule out this case from a practical point view. However, it is a property which is readily interpreted in practice because of the form of (7). Simply, we may rewrite (7) in the form

$$X_n = \begin{cases} KX_{n-1} & \text{with probability } p \\ K \min(X_{n-1}, Z_n^*) & \text{with probability } (1-p) \end{cases} \quad (11)$$

and the "new" innovation r.v.  $Z_n^*$  is simply  $Z_n$  conditional on  $Z_n < \infty$ . Thus,

$$S_{Z^*}(x) = \frac{S_Z(x) - p}{1 - p}. \quad (12)$$

In addition, although the form of (11) is different, of the process to be discussed below can be derived similarly provided we work in terms of  $S_Z(x) = p + (1 - p)S_{Z^*}(x)$ . Note that from (11) we can see that sample paths of the process  $\{X_n\}$  will tend to exhibit a “runs up” type of behavior. This type of behavior is characteristic of, for example, river flow data, but the geometric increase implied by (11) may be too severe for general use.

### 3 Bivariate Distributions and Autocorrelations

Using (7) and (8) it is straightforward to show that the bivariate distribution of any two values in the process has survivor function

$$\begin{aligned} S_{X_n, X_{n-j}}(y, x) &= P(X_n > y, X_{n-j} > x) \\ &= P\{X_{n-1} > y/K, Z_n > y/K, X_{n-j} > x\} \end{aligned}$$

from (7). By repeated use of (7) we get

$$\begin{aligned} S_{X_n, X_{n-j}}(y, x) &= S_X \left( \max \left( x, y/K^j \right) \right) S_X(y)/S_X \left( y/K^j \right) \\ &= \begin{cases} S_X(x)S_X(y)/S_X \left( y/K^j \right) & y \leq K^j x \\ S_X(y) & y > K^j x \end{cases} \end{aligned} \quad (13)$$

Note also that this is not an absolutely continuous distribution because, from (3), there will be a non-zero value for

$$\begin{aligned} p_j &= P(X_n = K^j X_{n-j}) \\ &= P(Z_1 > X_0, Z_2 > KX_0, \dots, Z_j > K^{j-1}X_0) \end{aligned} \quad (14)$$

which may be evaluated as

$$\int_0^\infty \left\{ \frac{S_X(K^j x)}{S_X(x)} \right\} f_X(x) dx. \quad (15)$$

Hence, the bivariate distribution has probability  $p_j$  on the line  $X_n = K^j X_{n-j}$  and probability  $(1 - p_j)$  spread over the region defined by  $X_n < K^j X_{n-j}$  with survivor function given by (13). It is important to note that since the process

is Markov all distributional behavior is characterized by such bivariate distributions, especially the form for contiguous observations, i.e.  $k = 1$ .

In addition, note that the bivariate distributions of  $(X_n, X_{n-1})$  and  $(X_n, X_{n-j})$  differ only in that  $K$  in the former becomes  $K^j$  in the latter. Thus, the bivariate distribution of  $(X_n, X_{n-j})$  and its properties are easily derived from those of  $(X_n, X_{n-1})$  by replacing  $K$  by  $K^j$ . In particular, we are interested in the autocorrelation function of the process  $\{X_n\}$ , i.e.  $\rho_X(j) = \text{corr}(X_n, X_{n-j})$ ,  $j = 0, 1, \dots$ . Thus, if  $\rho_X(1) = r(K)$  then  $\rho_X(j) = r(K^j)$ ,  $j = 0, 1, \dots$ . This is a useful property since we can now derive the autocorrelation function for any lag  $j$  from  $\rho_X(1)$  alone.

Tavares (1980) claims to show that in the case where  $\{X_n\}$  defined in (7) has a negative exponential distribution the autocorrelation function of lag  $j$  is given by  $\rho_X(j) = p_j$ , where  $p_j$  is defined by (14). In this negative exponential case,  $p_j = (1/K)^j$ . Sim (1986) uses exactly the same argument in the case when  $\{X_n\}$  is marginally Weibull and again derives  $\rho_X(j) = (1/K)^j$ . These results are of particular interest since this geometric autocorrelation function is associated with the well known autoregressive process of order one, the AR(1).

Unfortunately, although the autocorrelation result is true for the negative exponential case (as may be seen from Chernick, Daley, Littlejohn, 1988), the proof indicated by Tavares does not appear to be valid and certainly does not extend to the Weibull case, as Sim states. In general,  $\rho_X(j) \neq p_j$  although equality holds in the negative exponential case. By considering  $E(X_n | X_{n-1})$ , we may show that for the general stationary process defined by (7)

$$E(X_n X_{n-1}) = K E \left\{ X \int_0^X S_Z(z) dz \right\}. \quad (16)$$

Using  $E \left\{ K \int_0^X S_Z(z) dz \right\} = E(X) = m_X$ , we may extend (16) to obtain an expression for the covariance thus:

$$C(X_n, X_{n-1}) = K E \left\{ (X - m_X) \int_0^X S_Z(z) dz \right\}. \quad (17)$$

Hence,  $\rho_X(1)$  may be obtained from (17) by dividing by  $\text{Var}(X_n)$ , and  $\rho_X(j)$  may be obtained by replacing  $K$  in  $\rho_X(1)$  by  $K^j$ . (In this context, recall that  $S_Z(z)$  is a function of  $K$ ). The geometric autocorrelation for the negative exponential case claimed by Tavares may be readily shown from (17). In passing, we note that the autocorrelation function has this geometric form in general if and only if  $\rho_X(1) = (1/K)^\alpha$  for some  $\alpha > 0$ . The quantity  $p_j$  defined by (14) is also a useful measure of dependence, although we do not consider it in any detail here. The sequence  $\{p_j : j = 0, 1, 2, \dots\}$  is the survivor function of the length,  $T$  say, of runs of the form  $\{X, KX, K^2X, \dots, K^T X\}$ , and so  $P(T \geq j) = p_j$ .

## 4 Examples of Minification Processes

- (i) A simple example of a distribution which cannot be the marginal distribution for a minification process as defined by (7) is given by taking  $X$  to be uniformly distributed over  $(0,1)$  with probability 0.5 and uniformly distributed over  $(1,6)$  with probability 0.5. In this case

$$S_X(x) = \begin{cases} 1 - 0.5x & 0 < x < 1 \\ 0.6 - 0.1x & 1 < x < 6 \end{cases}$$

and it is a matter of straightforward calculation to show that if  $1 < K < 3$  then  $S_Z$  as defined by (8) is increasing for all  $x$  in the interval  $(1/K, 1)$ . We may also verify that the condition given by (10) is violated.

- (ii) An important but simple example of a minification process is provided by the uniform distribution on  $(0,1)$ . In this case,  $S_X(x) = 1 - x, 0 < x < 1$ , and  $x\lambda_X(x) = x/(1 - x)$  is clearly increasing so that (10) is satisfied. Further, from (8)

$$S_Z(z) = \frac{1 - Kz}{1 - z}, \quad 0 < z < \frac{1}{K} \quad (18)$$

which is a proper survivor function and  $Z$  is given by  $Z = U/(K - 1 + U)$ , where  $U$  is uniform on  $(0,1)$ . In addition, the autocorrelation function

of  $\{X_n\}$  can be derived from (11) and is found to be  $\rho_X(1) = 1/K$ , so that  $\rho_X(j) = (1/K)^j$ ,  $j = 0, 1, \dots$ . Thus, this uniform process enjoys the geometrically decaying autocorrelation of the AR(1) process. Processes which are marginally uniform are important since the uniform random variable  $X_n$  can be given any other distribution by means of the inverse distribution function transformation. Thus, if we wish a random variable  $Y$  with distribution functions  $F_Y(y)$ , we use  $Y = F_Y^{-1}(X)$ . This idea will be discussed in more detail later.

(iii) It is clear from (10) that  $X_n$  defined by (7) may have any marginal distribution whose hazard rate is itself non-decreasing, e.g. the uniform above, a Gamma  $(\kappa, \beta)$  with  $\kappa \geq 1$  or a Weibull  $(\kappa, \beta)$  with  $\kappa \geq 1$ . We consider now a distribution whose hazard decreases over the sample space, the Weibull  $(\kappa, \beta)$  with  $\kappa < 1$ . This is detailed in Sim (1986), but we note here that  $x\lambda(x) = \kappa\beta^\kappa x^\kappa$  is increasing for all  $\kappa$ , and so condition (10) is satisfied. Further,  $S_Z(x) = \exp\{-(K^\kappa - 1)(\beta x)^\kappa\} \rightarrow 0$  as  $x \rightarrow \infty$ . Thus, the general minification process (7) accommodates the Weibull distribution for all  $\kappa > 0$ . However, as noted above, Sim's derivation of the autocorrelation is wrong. He shows, correctly in this case, that  $p_j = (1/K)^j$ , but  $\rho_X(j) \neq p_j$  in general. Using (17), it is possible to show that the form of  $\rho_X(1)$  for general  $K$  cannot easily be derived. The case when  $\kappa = 2$  is tractable, however, and in this case

$$\rho_X(1) = \left\{ \frac{K}{2\sqrt{K^2 - 1}} \cos^{-1} \left( \frac{1}{K} \right) + \frac{1}{2K} - \frac{\pi}{4} \right\} / \left( 1 - \frac{\pi}{4} \right)$$

(iv) The Pareto distribution provides an example in which the form (11) is required rather than (7). Here  $\lambda_X(x) = \alpha/(1+x)$  with  $\alpha > 0$  a shape parameter, and condition (10) is again satisfied. However,  $S_Z(x) = \{(1+x) / (1+Kx)\}^\alpha \rightarrow K^{-\alpha}$  as  $x \rightarrow \infty$ . Hence, the process  $\{X_n\}$  with this Pareto marginal distribution may be generated using (11) and (12) with

$p = K^{-\alpha}$  and an innovation process  $\{Z_n\}$  whose survivor function is

$$\{[K(1+x)/(1+Kx)]^\alpha - 1\} / (K^\alpha - 1).$$

Again the autocorrelation function can be derived from (17) and we find that, for  $\alpha > 2$ ,  $\rho_X(j) = (1/K)^j$ ,  $j = 0, 1, \dots$ . Thus, the Pareto, like the uniform minification process, has the familiar geometric autocorrelations of the AR(1) process.

(v) An interesting case is that of the so-called bathtub hazard rate (Gaver and Acar, 1979), which we could model as  $\lambda(x) = p\alpha^p x^{p-1} + \beta + P\gamma^P x^{P-1}$  with  $0 < p < 1 < P$ . This models a situation where components have high likelihood of early, infant failure, otherwise have a constant hazard rate and then finally reach a “wear-out” state corresponding to the Weibull distribution with  $P > 1$ . This is actually the hazard rate of a random variable which is generated as the minimum of three independent random variables, one being exponential( $\beta$ ), the others being Weibull( $p, \alpha$ ) and Weibull( $P, \gamma$ ). Then  $x\lambda(x) = p\alpha^p x^p + \beta x + P\gamma^P x^P$ , which is clearly increasing in  $x$ , so that a minification process exists. In fact

$$\begin{aligned} \lambda_Z(x) &= K\lambda_X(Kx) - \lambda_X(x) = p \left\{ \alpha [K^p - 1]^{1/p} \right\}^p x^{p-1} \\ &\quad + \beta(K - 1) + P \left\{ \gamma [K^P - 1]^{1/P} \right\}^P x^{P-1}, \end{aligned}$$

showing that  $Z$  is again a random variable with a bathtub hazard rate and is easily generated as a minimum of two independent Weibull random variables and an independent exponential random variable.

The form given by Gaver and Acar (1979) uses a Pareto distribution for the early, decreasing hazard rate instead of the Weibull( $p, \alpha$ ) here. As we have seen in (iii) this will also be suitable for a minification process. Although, in both the cases considered here, the process is easy to generate the correlation structure is difficult to determine analytically.

## 5 Monotonic Transformation of the Minification Process

As noted in the introduction, one useful way to generalize the minification process given by (7) is to start with the negative exponential marginal case, Tavares (1980a, b), and take a monotonic transformation of each  $X_n$ . Thus suppose  $g$  is a monotone increasing function; then we define  $Y_n = g(X_n)$  and  $W_n = g(Z_n)$  for each  $n$ . Recall that if  $X_n$  is negative exponential of mean 1 then  $Z_n$  is also negative exponential and of mean  $(K-1)^{-1}$ .

It is straightforward to verify that the process  $\{Y_n\}$  is stationary and Markov and defined by

$$\begin{aligned} Y_n &= \min(g[Kg^{-1}(Y_{n-1})], g[Kg^{-1}(W_n)]) \\ &= g[Kg^{-1} \min(Y_{n-1}, W_n)]. \end{aligned} \tag{19}$$

If  $g$  is monotone decreasing then we must replace  $\min$  in (19) by  $\max$ .

Note that by definition  $X_n$  is negative exponential of unit mean so that if  $Y_n$  is to have cumulative hazard function  $\Lambda_Y(y)$  then

$$\exp(-\Lambda_Y(y)) = S_Y(y) = S_X(g^{-1}(y)) = e^{-g^{-1}(y)}$$

Hence

$$g^{-1}(y) = \Lambda_Y(y). \tag{20}$$

Note also that if  $g$  is decreasing then  $-g$  is increasing so we need consider only increasing transformations.

The bivariate distribution of any two observations may be obtained from (13) and is given by the joint survivor function

$$S_{Y_n, Y_{n-j}}(y, x) = \frac{S_Y \left[ \max \left\{ x, g \left( g^{-1}(y)/K^j \right) \right\} \right] S_Y(y)}{S_Y(g(g^{-1}(y)/K^j))} \tag{21}$$

Again, this bivariate distribution is mixed with probability given by (14), i.e.  $p_j = P(g^{-1}(Y_n) = K^j g^{-1}(Y_{n-j}))$  distributed on the curve  $Y_n = g[K^j g^{-1}(Y_{n-j})]$ ,

and the remaining  $(1 - p_j)$  distributed over  $Y_n < g \left[ K^j g^{-1} (Y_{n-j}) \right]$  by means of the survivor function ( 21). When  $g$  is monotonic decreasing we replace survivor functions by distribution functions in ( 21).

We can derive the autocorrelation function for  $\{Y_n\}$  from first principles in the same way as for  $\{X_n\}$ . However, we can also relate the moments of the transformed series to those of the original negative exponential. For example,

$$E(Y_n Y_{n-1}) = E \left\{ g(X) \left[ g(KX) S_Z(X) + \int_0^X g(Kz) f_Z(z) dz \right] \right\} \quad (22)$$

which, when  $g(0)$  is finite, can be simplified to

$$E(Y_n Y_{n-1}) = E \left\{ g(X) \left[ g(0) + K \int_0^X g'(Kz) S_Z(z) dz \right] \right\} \quad (23)$$

where  $X$  is negative exponential of unit mean, and  $S_Z(z) = e^{-(K-1)z}$ .

## 6 Examples of the Transformation Process

- (i)  $Y_n = -\ln X_n$  so that  $Y_n$  has the extreme value or Gumbel (1) distribution with distribution function  $e^{-e^{-x}}$ . Note that  $g$  is monotonic decreasing and so the process is defined by

$$Y_n = \max(Y_{n-1}, W_n) + b,$$

where  $b = -\ln K$ . In addition, the innovation process  $\{W_n\}$  is also an extreme value random variable. This is exactly the process introduced by Tavares (1977) and examined in some hydrological contexts. He was unable to specify the autocorrelation function then but noted it appeared to be exponential. An examination of ( 22) suggest it would be very difficult to obtain in closed form.

(ii)  $Y_n = X_n^{1/\kappa}, \kappa > 0$ , so that  $Y_n$  is Weibull with parameter  $\kappa$ . We find that the process is defined by ( 16), i.e.

$$Y_n = K^{1/\kappa} \min(Y_{n-1}, W_n)$$

This is in effect the Weibull process discussed by Sim (1986), and in an earlier example in this paper.

(iii)  $Y_n = X_n^{-1/\kappa}, \kappa > 0$ , so that  $Y_n$  is the second type of extreme value distribution (Johnson and Kotz, 1970, Ch.21) with distribution function  $F_Y(y) = \exp(-y^{-\kappa}), y > 0$ . The process is defined by

$$Y_n = K^{-\kappa} \max(Y_{n-1}, W_n).$$

but again the autocorrelation appears unobtainable in closed form.

(iv)  $Y_n = 1 - e^{-X_n}$ , so that  $Y_n$  is now uniformly distributed on (0,1). Here, we have  $g^{-1}(y) = -\ln(1 - y)$  and the transformation is an increasing one, so that (9) may be used directly to show that

$$Y_n = \min(1 - (1 - Y_{n-1})^K, 1 - (1 - W_n)^K)$$

where  $W_n$  has survival function  $S_W(w) = (1 - w)^{K-1}$ . The formulation may be simplified by using the decreasing transformation  $Y_n = e^{-X_n}$ , in which case we obtain

$$Y_n = \max(Y_{n-1}^K, U_n^{K/(K-1)}), \quad (24)$$

where  $U_n$  is uniformly distributed on (0,1). Both processes share the same autocorrelation function and application of ( 23) yields it in the form

$$\rho_X(j) = \frac{3}{2K^j + 1}. \quad (25)$$

(v) The uniform process of (iv) is important also because we may use it to generate any other marginal distribution by means of the inverse distribution function transformation. Thus, if  $Y$  is to have distribution function

$F_Y(X)$  we can generate the process using the transformation

$$Y_n = F_Y^{-1}(1 - e^{-X_n}). \quad (26)$$

Now,  $g^{-1}(x) = -\ln S_Y(x)$ , and we note that this is the cumulative hazard function of  $Y$ . Note that (26) is a monotone increasing transformation so that all the results (19) – (23) may be applied directly. In particular, note that if (23) is applicable it may be written in the form

$$E(Y_n Y_{n-1}) = E \left\{ g(X) \left[ g(0) + \int_{g(0)}^{g(KX)} (S_Y(y))^{1-\frac{1}{K}} dy \right] \right\}$$

where

$$g(x) = F_Y^{-1}(1 - e^{-x}).$$

Note also that the results (19) – (23) hold when  $\{Y_n\}$  is a transformed version of  $\{X_n\}$ , given by  $Y_n = g(X_n)$ , given only that  $\{X_n\}$  is a minification process satisfying (7). We have considered the case where  $\{X_n\}$  is the negative exponential process but, in fact,  $\{X_n\}$  may be any minification process and (19) - (23) still hold with obvious modifications to the comments immediately after (23). In particular, it may well be simpler to choose  $\{X_n\}$  to be uniform on  $(0,1)$  and then take  $g(x) = F_Y^{-1}(x)$ . We have specified two distinct uniform minimum processes in this paper and either may be used in this way, although the first, example (ii) of Section 4, is much simpler to implement.

As an example of such a procedure we consider the distribution of example (i) of Section 4. We noted that for  $K \in (1, 3)$  no minification process exists with this marginal distribution. We show here how to derive a suitably transformed process beginning with the uniform minification process of example (ii) in Section 4. Now,  $g(x) = F_Y^{-1}(x)$ , i.e.

$$g(x) = \begin{cases} 2x & 0 < x < 0.5 \\ 10x - 4 & 0.5 < x < 1. \end{cases}$$

For simplicity, we take  $K = 2$ , in which case  $g[Kg^{-1}(x)] = g(x)$  and so the process is given by

$$V_n = \min(Y_{n-1}, W_n)$$

with

$$Y_n = \begin{cases} 2V_n & 0 < V_n < 0.5 \\ 10V_n - 4 & 0.5 < V_n < 1, \end{cases}$$

where  $W_n = 2Z_n = 2U_n/(1 + U_n)$ , from example (ii) of Section 4, and  $U_n$  is uniform on  $(0,1)$ .

## 7 Maximum Processes

We may define a maximum process  $\{X_n\}$  in exactly the same way as the minification process of (7). We use

$$X_n = \alpha \max(X_{n-1}, Z_n) \quad 0 < \alpha < 1 \quad (27)$$

and now the analog of (8) replaces survival functions by distribution functions and an argument similar to the one about  $K$  shows  $\alpha \in (0, 1)$ . The discussion goes through in an exactly analogous fashion but is somewhat less elegant since there is no natural interpretation of the analogue of the hazard rate when using distribution functions. Nevertheless, most results are duplicated with survivor functions replaced by distribution function. For example, the analog of (17) is exactly the same with  $S_Z(z)$  replaced by  $F_Z(z) = P(Z \leq z)$ .

As a simple example we present the uniform maximum process. Since  $F_Z(z) = F_X(\alpha z)/F_X(z)$  it follows that

$$Z = \begin{cases} 0 & \text{with probability } \alpha \\ \alpha + (1 - \alpha)U & \text{with probability } 1 - \alpha \end{cases} \quad (28)$$

where  $U$  is uniform on  $(0,1)$ . We may also derive the autocorrelation function for this process and it is in the geometric form of an AR(1). In particular

$$\rho_X(j) = \alpha^{2j} \quad j = 0, 1, \dots \quad (29)$$

Note, however, that because of the special mixed form of  $Z_n$  we can rewrite the process (27) in the form

$$X_n = \begin{cases} \alpha X_{n-1} & \text{with probability } \alpha \\ \alpha + (1 - \alpha)U_n & \text{with probability } 1 - \alpha \end{cases} . \quad (30)$$

Obviously this result is specific to the uniform maximum but it serves to illustrate the differences which can arise between maximum and minimum processes.

## 8 Numerical Example: Three Uniform Processes

Three uniform autoregressive process have been derived in this paper. The first is the minification process (ii) of Section 4, with  $\rho_X(j) = (1/K)^j$ ,  $j = 0, 1, \dots$ . The second uniform process is the transformation process given as example (iv) of Section 6, with correlation given in (25) as  $\rho_X(1) = 3/(2K^j + 1)$ . The third uniform process given is the maximum process given in Section 7 with  $Z$  given at (28).

Sample paths are shown for these three processes in Figure 1. All three are generated from the same uniform i.i.d sequence  $U_n$ ,  $n = 0, 1, \dots, 100$ . Also all three have the same value for  $\rho_X(1)$ , namely 0.9. This means that for the minification process  $K = 1/\rho_X(1) = 1/(0.9)$ , but for the transformation process  $K = 7/6$ . For the third process  $\alpha = (0.9)^{0.5}$ , from (29). Note also that the marginal distributions are uniform and therefore bounded by zero and one, unlike most time series for which sample paths or data are exhibited.

In the top panel of Figure 1, we see the typical “runs up” or “run off” behavior of a minification process. Again the middle panel shows that the transformation process exhibits “runs down”, with the runs being convex down when they start at high values, and convex up when they start at low values. Note particularly the very slow decay at the end of the series when the process has very high values. In the third panel, the maximum uniform process shows very long geometric decays and has a very odd, persistent appearance.

Figure 2 shows the autocorrelation functions of the transformation and the minification uniform processes. The function  $\rho_X(j)$  for the minification process is less than the function  $\rho_X(j)$  for the transformation process. In fact, for large  $j$ ,  $\rho_X(j)$  for the transformation process has value approximately one and half times that of the autocorrelation  $\rho_X(j)$  for the minification process.

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## Uniform Processes

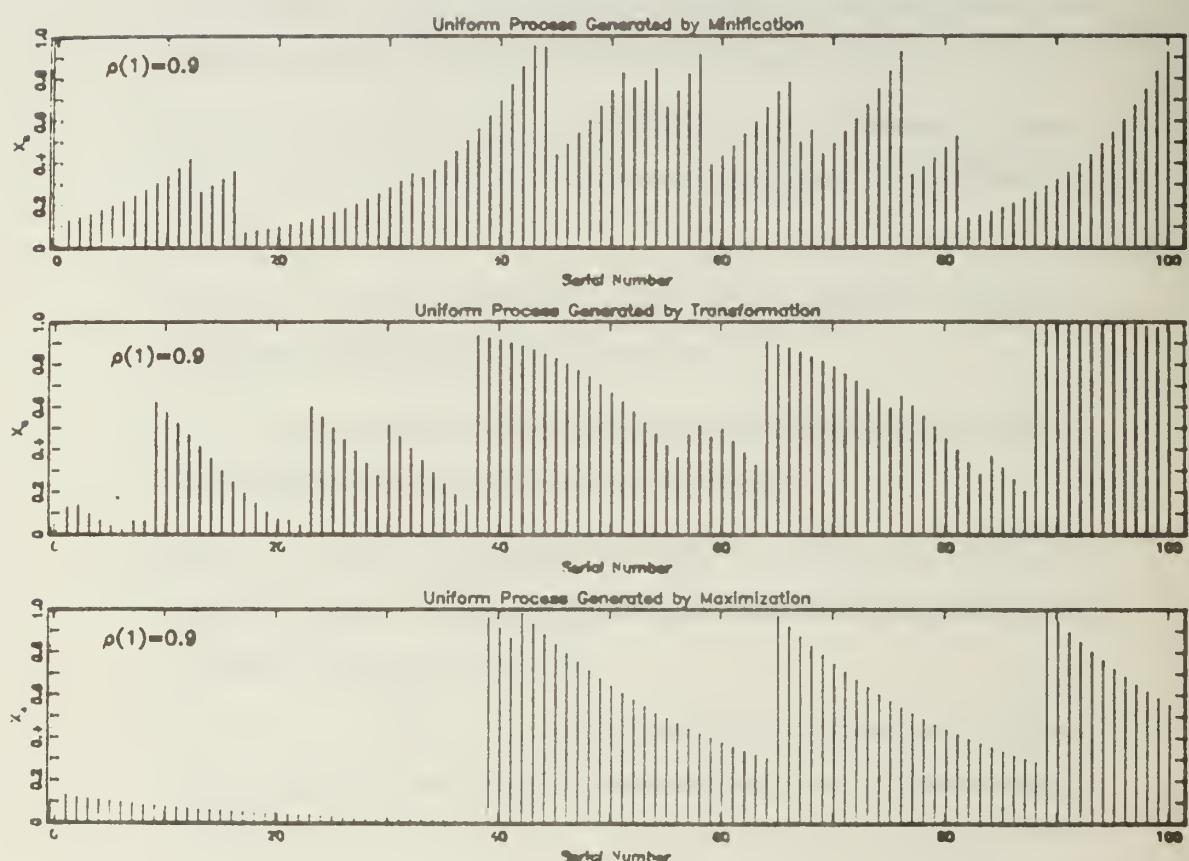


Figure 1: Sample paths for three processes with uniform marginal distributions.

**Uniform Processes**  
**Autocorrelation Functions for two Processes**

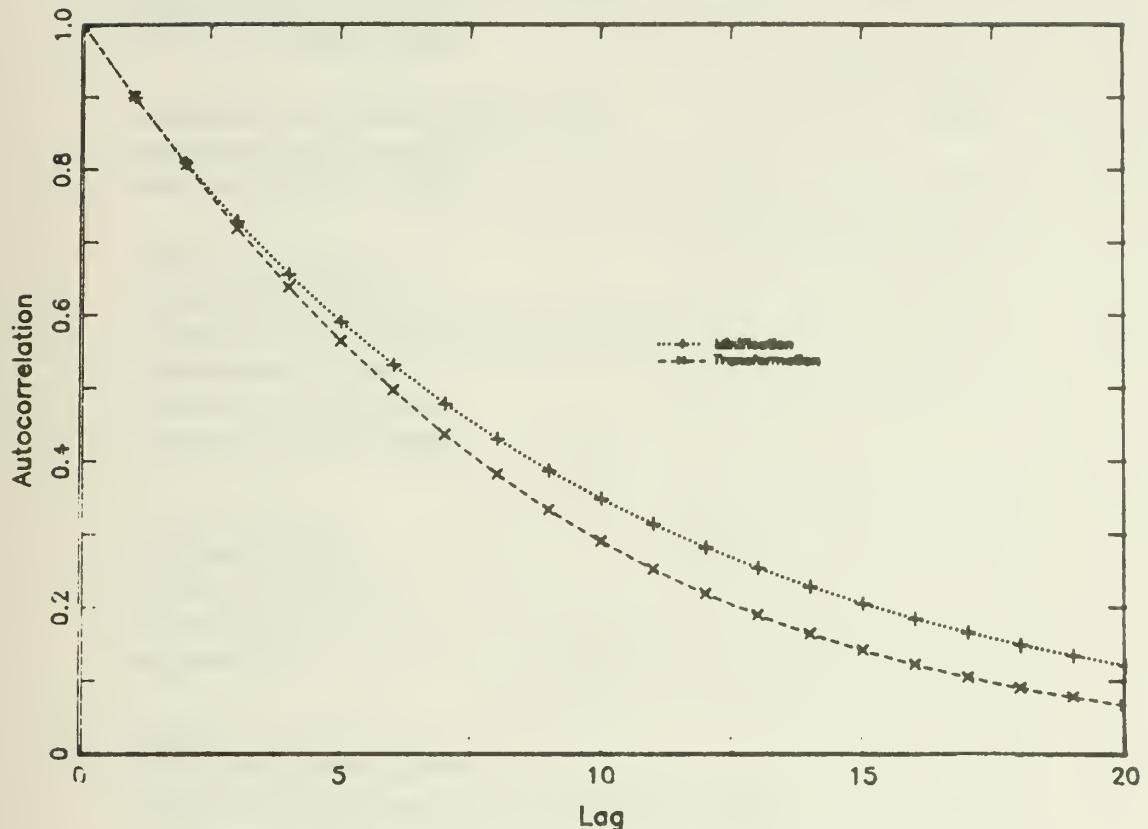


Figure 2: Correlation functions  $\rho_X(j)$  for the minification and transformation processes with uniform distribution.



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